Complex Variable

Theorem:

The continuous single valued function is will be analytic in a region R if and only if four partial derivatives exists, continuous and Cauchy Riemann equations , are satisfied.

Proof: If is analytic then





Exists and unique. Two cases arise.

Case 1. Along x-axis  and 



 ……………………………………………..(1)

Case 2. Along y-axis  and 





 ……………………………………………….(2)

From equation (1) and (2) we get



Therefore ,

Since and  are continuous then







 …………………………..(3)

Where  and  Since  and 

Similarly and  are continuous then



Where  and  Since  and 

Now 

Where  and  Since  and 

Now using Cauchy Riemann equations 









Dividing by  and Taking limit 



Hence  is analytic.

Problem: Show that the function  is harmonic. And find the conjugate harmonic function.

Problem: Show that the function  is harmonic. And find the conjugate harmonic function of 

Solution: Given that 





 and 

Therefore 

Implies that  is harmonic

From CR equations

 …………………. (1)

and  …………………..(2)

Integrating (1) on both sides 

 …………………(3)

Differentiate









Putting this value in equation (3)



Conjugate harmonic function

Problem: Show that  is harmonic. Also find the conjugate harmonic function of 

Problem: Find the conjugate harmonic function of 

Cauchy’s Integral formula:

If  is analytic for all points inside of C and connected a simple closed curve C.  is any point inside C. Then .

Proof: Since is analytic for all points inside of C

 where  , then 

Now .

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Considering C is very small so that for all points on C.

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Cauchy’s Integral formula for nth order derivative:

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Evaluate:

1.  where C is the circle
2.  where C is the circle
3.  where C is the circle
4.  where C is the circle
5.  where C is the circle 
6.  where C is the circle 

Solution 1:

We know

.

Here  and 

then 

Now 

Solution 2:

We know

.

Here  and 

then 

Now 

Solution 3:

We know

.

Now

 Here 



Solution 4.  where C is the circle

We know

.

 and 

then 



Solution 5.  where C is the circle 

We know





Now



Solution 6.  where C is the circle 

We know

.

,  and 

Now

 ……………………(1)

Here ,





From (1)



Problem 7. 

Problem 8. 

Theorem: if  is analytic inside and on a simple closed curve C except at the pole  of order  then the residue of  at  is 

If  is a simple pole then 

Cauchy’s Residue Theorem:

Let be analytic inside and on a simple closed curve C except at a finite number of singular points  then 

Proof: Let  be the center of the circle  respectively.  be analytic inside and on a simple closed curve C.

 (1)

But 



…………………………………………………………………………………………

………………………………………………………………………………………………………



From (1)





Ex: Show that 

Solution: let 

To find the pole, 

 is a pole of order 2.

Now

































By Cauchy residue theorem











